

Geometry I, 2006 : Coursework 3 Solutions

2. (i) [1 mark] Suppose $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. Then $\underline{u} \cdot \underline{v} = u_1v_1 + u_2v_2 + u_3v_3$.

(ii) [2 marks] $\underline{u} \cdot \underline{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \underline{v} \cdot \underline{u}$.

[Alternatively, this could be proved by noting that the definition of $\underline{u} \cdot \underline{v}$ gives the same value when the roles of $\underline{u}$ and $\underline{v}$ are interchanged.]

(iii) [1 mark] $\underline{u} \cdot (\alpha \underline{v}) = \underline{u} \cdot \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \end{pmatrix} = u_1\alpha v_1 + u_2\alpha v_2 + u_3\alpha v_3 = \alpha(u_1v_1 + u_2v_2 + u_3v_3) = \alpha(\underline{u} \cdot \underline{v})$.

[1 mark] $\alpha(\underline{u} \cdot \underline{v}) = \alpha(u_1v_1 + u_2v_2 + u_3v_3) = \alpha u_1v_1 + \alpha u_2v_2 + \alpha u_3v_3 = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \end{pmatrix} \cdot \underline{v} = (\alpha \underline{u}) \cdot \underline{v}$.

(iv) Applying (iii), we have:

[1 mark] $\underline{u} \cdot (-\underline{v}) = \underline{u} \cdot ((-1)\underline{v}) = (-1)(\underline{u} \cdot \underline{v}) = -(\underline{u} \cdot \underline{v})$, and

[1 mark] $-(\underline{u} \cdot \underline{v}) = (-1)(\underline{u} \cdot \underline{v}) = ((-1)\underline{u}) \cdot \underline{v} = (-\underline{u}) \cdot \underline{v}$.

3. (i) [1 mark; some calculation must be shown]

Let θ be the angle between $\underline{p}$ and $\underline{q}$. Then $\cos \theta = \underline{p} \cdot \underline{q} / (|\underline{p}||\underline{q}|) = 2/(\sqrt{6}\sqrt{10}) = 1/\sqrt{15}$.

(ii) [1 mark; some calculation must be shown]

$$((3\underline{p} + 2\underline{q}) \cdot (5\underline{p} - 3\underline{q}))(\underline{p} + \underline{q}) = \begin{pmatrix} 9 \\ 6 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 10 \\ -8 \end{pmatrix} (\underline{p} + \underline{q}) = 32 \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 128 \\ 64 \\ 0 \end{pmatrix}.$$

4. (i) [1 mark] $\begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 16$, and so a Cartesian equation for the plane is

$$2x - 3y + 6z = 16.$$

(ii) [1 mark; justification must be given] The point $(5, 0, 1)$ is on this plane since

$$2 \times 5 - 3 \times 0 + 6 \times 1 = 16.$$

(iii) [1 mark; some calculation must be shown] This distance is

$$\left| \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} - 16 \right| / \left| \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \right| = |20 - 16|/7 = 4/7.$$

5. (i) [1 mark iff all three are correct]

$$\underline{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \underline{q} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \underline{r} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

(ii) [1 mark each; some calculation or justification must be shown]

$$\underline{a} = \underline{q} - \underline{p} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}, \underline{b} = \underline{r} - \underline{p} = \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix}.$$

(iii) [1 mark; some justification must be given] $\underline{a}$ is not parallel to $\underline{b}$, since it is not a scalar multiple of $\underline{b}$.

(iv) [2 marks]

One solution is $\underline{n} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$

(v) [2 marks; some explanation or calculation must be given for full marks; one mark if only a (correct) equation for the plane is given]

The vector $\underline{n}$ in part (iv) is orthogonal to both of the non-zero non-parallel vectors $\underline{a}$ and $\underline{b}$, and so $\underline{n}$ is orthogonal to the (unique) plane containing the points P, Q, R .

Since $\underline{p} \cdot \underline{n} = 2 (= \underline{q} \cdot \underline{n} = \underline{r} \cdot \underline{n})$, a Cartesian equation for this plane is

$$x + 2y + z = 2.$$